

ON A QUESTION OF GLASBY, PRAEGER, AND XIA IN CHARACTERISTIC 2

MICHAEL J. J. BARRY

ABSTRACT. Recently, Glasby, Praeger, and Xia asked for necessary and sufficient conditions for the ‘Jordan partition’ $\lambda(m, n, p)$ to be standard. Previously we gave such conditions when p is any odd prime. Here we give such conditions when $p = 2$. Our main result is that $\lambda(m, n, 2)$ is never standard for $4 \leq m \leq n$.

For positive integers m and n with $m \leq n$, the **Jordan partition** $\lambda(m, n, p) = (\lambda_1, \dots, \lambda_m)$ is a partition of $m \cdot n$ into m parts that arises when writing the tensor product of two Jordan matrices over a field of characteristic p as a direct sum of Jordan matrices. See [3] for more background on this subject. The Jordan partition $\lambda(m, n, p)$ is **standard** iff $\lambda_i = m + n - 2i + 1$ for $i = 1, \dots, m$.

In response to a question asked in [3], we gave necessary and sufficient conditions for a Jordan partition $\lambda(m, n, p)$ to be standard when p is an odd prime in [1]. Here we handle the case of $p = 2$.

Theorem 1. *Let m and n be positive integers with $m \leq n$. Then $\lambda(m, n, 2)$ is standard iff one of the following three conditions holds:*

- (1) $m = 1$
- (2) $m = 2$ and n is odd
- (3) $m = 3$ and $n = 6 + 4k$ for a nonnegative integer k

Our main tool in establishing this result is the composition of m that we associate to $\lambda(m, n, p)$. For now we will allow p to be any prime.

Definition 1. Suppose that the Jordan partition

$$\lambda(m, n, p) = (\overbrace{\lambda_1, \dots, \lambda_1}^{m_1}, \overbrace{\lambda_2, \dots, \lambda_2}^{m_2}, \dots, \overbrace{\lambda_r, \dots, \lambda_r}^{m_r}) = (m_1 \cdot \lambda_1, \dots, m_r \cdot \lambda_r)$$

where $\lambda_1 > \dots > \lambda_r > 0$ and $\sum_{i=1}^r m_i = m$. Denote the composition (m_1, \dots, m_r) of m by $c(m, n, p)$.

Note that if $\lambda(m, n, p)$ is standard, then $c(m, n, p) = (m \cdot 1)$. The converse is true. But in fact much more is true: $\lambda(m, n, p)$ is completely determined by n and $c(m, n, p)$, as we will prove in Proposition 1.

We now record two properties of $c(m, n, p)$ which follow from Theorem 4 of [3].

Theorem 2. *Suppose that $m \leq p^t$.*

- (1) *Then $c(m, n, p) = c(m, n + p^t, p)$ for every integer $n \geq m$.*
- (2) *Then $c(m, p^t + i, p) = r(c(m, 2p^t - i, p))$ for every integer i in $[0, p^t]$.*

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In [2], we gave a recursive definition of the sequence $s_p(m, n)$, where $m \leq n$, of $m + n$ integers in six mutually exclusive and exhaustive cases. In [1], we related it to the Jordan partition $\lambda(m, n, p)$ by noting that

$$\lambda(m, n, p) = (s_p(m, n)(1), \dots, s_p(m, n)(m)),$$

that is, $\lambda(m, n, p)$ is the subsequence of the first m elements of $s_p(m, n)$. In light of the definition of $s_p(m, n)$, $c(m, n, p)$ satisfies the following conditions in the corresponding cases:

- (1) $c(m, n, p) = (m + n - p^{k+1}) \oplus c(p^{k+1} - n, p^{k+1} - m, p)$
- (2) $c(m, n, p) = (c + d - p^k) \oplus c((a + b + 1)p^k - n, (a + b + 1)p^k - m, p)$
- (3) $c(m, n, p) = c_1 \oplus c((a + b)p^k - n, (a + b)p^k - m, p)$ where
 $c_1 = c(\min(c, d), \max(c, d), p) \oplus (|c - d|) \oplus r(c(\min(c, d), \max(c, d), p))$
- (4) $c(m, n, p) = c(m, bp^k + d, p) = r(c(m, bp^k - d, p))$
- (5) $c(m, n, p) = c(c, bp^k, p) = (m)$
- (6) $c(m, n, p) = c(ap^k, bp^k, p) = (p^k) \oplus c((a - 1)p^k, (b - 1)p^k, p)$

where \oplus is the operation of concatenating two sequences into one and $r(s)$ denotes the reverse of the sequence s . See [1] or [2] for more details of the six cases.

Of course $c(m, n, p)$ determines m since $\sum_{i=1}^r m_i = m$. Next we show that n and $c(m, n, p)$ determine $\lambda(m, n, p)$.

Proposition 1. *If $c(m, n, p) = (m_1, \dots, m_r)$, then*

$$\lambda_i = \frac{1}{m_i} \sum_{k=m_1+\dots+m_{i-1}+1}^{m_1+\dots+m_i} (m + n - 2k + 1) = n + \sum_{k=i+1}^r m_k - \sum_{k=1}^{i-1} m_k$$

when $1 \leq i \leq r$.

Proof. Note that

$$n + \sum_{k=i+1}^r m_k - \sum_{k=1}^{i-1} m_k = m + n - 2 \sum_{k=1}^{i-1} m_k,$$

and we will use the second form at points of the proof.

Let k be the unique nonnegative integer such that $p^k \leq n < p^{k+1}$. Write $n = bp^k + d$ with $0 < b < p$ and $0 \leq d < p^k$. Write $m = ap^k + c$ with $0 \leq a < p$ and $0 \leq c < p^k$. Note that $a + c > 0$.

We proceed by induction on $m + n$. The result is true for $(m, n) = (1, 1)$. Assume the result is true for all (m', n') with $m' + n' < m + n$.

Case 1: $m + n > p^{k+1}$. Then $m_1 = m + n - p^{k+1}$, $\lambda_1 = p^{k+1}$, and $c(p^{k+1} - n, p^{k+1} - m, p) = (m_2, \dots, m_r)$. Note that $n + m - m_1 = p^{k+1} = \lambda_1$, that is, the formula holds for $i = 1$. By inductive assumption, when $2 \leq i \leq r$,

$$\begin{aligned} \lambda_i &= (p^{k+1} - m) + (m_{i+1} + \dots + m_r) - (m_2 + \dots + m_{i-1}) \\ &= (n - m_1) + (m_{i+1} + \dots + m_r) - (m_2 + \dots + m_{i-1}) \\ &= n + (m_{i+1} + \dots + m_r) - (m_1 + m_2 + \dots + m_{i-1}), \end{aligned}$$

showing the formula holds $i = 2, \dots, r$.

Case 2: $m + n \leq p^{k+1}$ but $c + d > p^k$. Then $m_1 = c + d - p^k$, $\lambda_1 = (a + b + 1)p^k$, and $c((a + b + 1)p^k - n, (a + b + 1)p^k - m, p) = (m_2, \dots, m_r)$. Note that $n + m - m_1 = (a + b + 1)p^k = \lambda_1$, that is, the formula holds for $i = 1$. Verification that the formula for λ_i holds when $2 \leq i \leq m$ is similar to Case 1 using the fact that $(a + b + 1)p^k - m = n - m_1$.

Case 3: $m + n \leq p^{k+1}$, $1 \leq c + d \leq p^k$, and $a > 0$. Suppose that

$$c(\min(c, d), \max(c, d), p) = (m'_1, \dots, m'_s)$$

and

$$\lambda(\min(c, d), \max(c, d), p) = (m'_1 \cdot \lambda'_1, \dots, m'_s \cdot \lambda'_s).$$

So $m'_i = m_i$ and $\lambda'_i = \lambda_i - (a + b)p^k$ when $1 \leq i \leq s$. Let $t = 2s + \epsilon$ where $\epsilon = 1$ if $c \neq d$ and $\epsilon = 0$ if $c = d$. Then $c((a + b)p^k - n, (a + b)p^k - m, p) = (m_{t+1}, \dots, m_r)$ and

$$(m_1, \dots, m_t) = \begin{cases} (m'_1, \dots, m'_s, m'_s, \dots, m'_1), & \text{if } c = d \\ (m'_1, \dots, m'_s, |c - d|, m'_s, \dots, m'_1), & \text{if } c \neq d \end{cases}$$

By inductive assumption, when $1 \leq i \leq s$,

$$\lambda'_i = c + d - 2 \sum_{k=1}^{i-1} m'_k - m'_i = c + d - 2 \sum_{k=1}^{i-1} m_k - m_i.$$

Thus, when $1 \leq i \leq s$,

$$\lambda_i = \lambda'_i + (a + b)p^k = m + n - 2 \sum_{k=1}^{i-1} m_k - m_i$$

and the formula holds.

Assume that $c \neq d$, so $\lambda_{s+1} = 0 + (a + b)p^k$ and $m_{s+1} = |c - d|$. Now

$$\begin{aligned} \lambda_{s+1} &= (a + b)p^k \\ &= (a + b)p^k + c + d - 2 \min(c, d) - |c - d| \\ &= m + n - 2 \sum_{k=1}^s m_k - m_{s+1} \end{aligned}$$

and the formula holds for $i = s + 1$.

We now consider the case of $t - s + 1 \leq i \leq r$. Then $m_i = m_{t+1-i}$ and $\lambda_i = -\lambda'_{t+1-i} + (a + b)p^k$. Thus

$$\begin{aligned} \lambda_i &= -\lambda'_{t+1-i} + (a + b)p^k \\ &= (a + b)p^k - (c + d) + 2 \sum_{k=1}^{t-i} m_k + m_{t-i+1} \\ &= m + n - 2(c + d) + 2 \sum_{k=1}^{t-i} m_k + m_{t-i+1} \\ &= m + n - 2 \min(c, d) - |c - d| - (c + d) + 2 \sum_{k=1}^{t-i} m_k + m_{t-i+1} \\ &= m + n - 2 \sum_{k=1}^s m_i - |c - d| - (c + d) + 2 \sum_{k=1}^{t-i} m_k + m_{t-i+1} \\ &= m + n - 2 \sum_{k=1}^s m_i - |c - d| - |c - d| - 2 \sum_{k=1}^s m_k + 2 \sum_{k=1}^{t-i} m_k + m_{t-i+1} \\ &= m + n - 2 \sum_{k=1}^s m_i - |c - d| - |c - d| - 2 \sum_{k=t-i+1}^s m_k + m_{t-i+1} \end{aligned}$$

$$\begin{aligned}
&= m + n - 2 \sum_{k=1}^s m_i - 2|c - d| - 2 \sum_{k=t-s+1}^i m_k + m_i \\
&= m + n - 2 \sum_{k=1}^s m_i - 2|c - d| - 2 \sum_{k=t-s+1}^{i-1} m_k - m_i \\
&= m + n - 2 \sum_{k=1}^{i-1} m_k - m_i.
\end{aligned}$$

By inductive assumption, when $t + 1 \leq i \leq m$,

$$\begin{aligned}
\lambda_i &= (a + b)p^k - m + (m_{i+1} + \cdots + m_r) - (m_{t+1} + \cdots + m_{i-1}) \\
&= bp^k - c + (m_{i+1} + \cdots + m_r) - (m_{t+1} + \cdots + m_{i-1}) \\
&= bp^k + d + (m_{i+1} + \cdots + m_r) - (c + d) - (m_{t+1} + \cdots + m_{i-1}) \\
&= n + (m_{i+1} + \cdots + m_r) - (2 \min(c, d) + |c - d|) - (m_{t+1} + \cdots + m_{i-1}) \\
&= n + (m_{i+1} + \cdots + m_r) - (m_1 + \cdots + m_t) - (m_{t+1} + \cdots + m_{i-1}).
\end{aligned}$$

Thus the formula for λ_i holds when $t + 1 \leq i \leq r$.

Case 4: $m + n \leq p^{k+1}$, $1 \leq c + d \leq p^k$, $a = 0$ (so $m = c$), and $d > 0$. Then $c(m, n, p) = r(c(m, bp^k - d, p))$. Suppose that $c(m, bp^k - d, p) = (m'_1, \dots, m'_r)$ and $\lambda(m, bp^k - d, p) = (m'_1 \cdot \lambda'_1, \dots, m'_r \cdot \lambda'_r)$. Hence $m_i = m'_{r+1-i}$ and $\lambda_i = \lambda'_{r+1-i} + 2bp^k$. Then, by inductive assumption,

$$\lambda'_i = (bp^k - d) + (m'_{i+1} + \cdots + m'_r) - (m'_1 + \cdots + m'_{i-1})$$

and so

$$\begin{aligned}
\lambda'_{r+1-i} &= (bp^k - d) + (m'_{r+2-i} + \cdots + m'_r) - (m'_1 + \cdots + m'_{r-i}) \\
&= (bp^k - d) + (m_{i-1} + \cdots + m_1) - (m_r + \cdots + m_{i+1}).
\end{aligned}$$

Therefore

$$\begin{aligned}
\lambda_i &= -\lambda'_{r+1-i} + 2bp^k \\
&= 2bp^k - bp^k + d - (m_{i-1} + \cdots + m_1) + (m_r + \cdots + m_{i+1}) \\
&= n + (m_{i+1} + \cdots + m_r) - (m_1 + \cdots + m_{i-1})
\end{aligned}$$

when $1 \leq i \leq r$.

Case 5: $m + n \leq p^{k+1}$, $1 \leq c + d \leq p^k$, $a = 0$, and $d = 0$, so $(m, n) = (c, bp^k)$. In this case $r = 1$, $m_1 = m$, and $\lambda_1 = n$. Therefore $n + m - m_1 = \lambda_1$ and the formula is valid.

Case 6: $m + n \leq p^{k+1}$, $c = d = 0$, so $0 < a < a + b \leq p$. Then $m_1 = p^k$, $\lambda_1 = (a + b - 1)p^k$, and $c((a - 1)p^k, (b - 1)p^k, p) = (m_2, \dots, m_r)$. Because $n + m - m_1 = \lambda_1$, the formula holds for $i = 1$. Verification that the formula for λ_i holds when $2 \leq i \leq m$ is similar to Case 1. \square

Corollary 1. *If m and n are positive integers with $m \leq n$, then $\lambda(m, n, p)$ is standard iff $c(m, n, p) = (m \cdot 1)$.*

Proof of Theorem 1. It is clear that $c(1, n, 2) = (1)$ and so $\lambda(1, n, 2)$ is standard for all integers $n \geq 1$. Suppose that $2^{t-1} < m \leq 2^t$. By the periodicity result of Theorem 2, it suffices to compute $c(m, n, 2)$ for integers n in the interval $[2^t, 2^{t+1} - 1]$.

First consider the case of $m = 2^t$ where t is a positive integer. Then, by Case 6, $c(m, 2^t) = (2^t)$. Now we look at the case $(m, n) = (2^t, 2^t + i)$, where $1 \leq i \leq 2^t - 1$. Then we are in Case 1, so $c(m, n, 2) = (i) \oplus c(2^t - i, 2^t, 2)$. Since $(2^t - i, 2^t, 2)$ is a Case 5 situation, $c(2^t - i, 2^t, 2) = (2^t - i)$. Thus $c(m, n, 2) = (i) \oplus (2^t - i)$. We have shown that $c(2, n, 2) = (1, 1)$ iff n is odd, and that $c(4, n, 2)$ never equals $(1, 1, 1, 1)$. By Corollary 1, $\lambda(2, n, 2)$ is standard iff n is odd.

Next we prove by induction on m that $c(m, n, p)$ never equals $(m \cdot 1)$ for $4 \leq m \leq n$. We have already verified this for the base case $m = 4$ and any $m = 2^t \geq 4$. We can assume that $4 \leq 2^{t-1} < m < 2^t$, and write $m = 2^{t-1} + j$ where $1 \leq j \leq 2^{t-1} - 1$. Then $(m, 2^t)$ is a Case 5 situation, so $c(m, 2^t, 2) = (m) \neq (m \cdot 1)$. When $1 \leq i \leq 2^{t-1} - j$, $(m, 2^t + i)$ is a Case 4 situation, and so $c(m, 2^t + i, 2) = r(c(m, 2^t - i, 2))$. By our inductive assumption, $c(m, 2^t - i, 2) \neq (m \cdot 1)$, which implies that $c(m, n, 2) \neq (m \cdot 1)$.

When $2^{t-1} - j + 1 \leq i \leq 2^t - 1$, $(m, 2^t + i)$ is a Case 1 situation. Therefore $c(m, 2^t + i, 2) = (m + i - 2^t) \oplus c(2^t - i, 2^t + 2^{t-1} - j, 2)$. If $m + i - 2^t > 1$, then $c(m, 2^t - i, 2) \neq (m \cdot 1)$. Assume that $m + i - 2^t = 1$. Then $2^t - i = m - 1 \geq 4$. By inductive assumption, $c(2^t - i, 2^t + 2^{t-1} - j, 2) \neq ((m - 1) \cdot 1)$, which implies that $c(m, 2^t + i, 2) \neq (m \cdot 1)$. This completes our proof by induction.

Finally one can check that $c(3, 4, 2) = (3)$, $c(3, 5, 2) = (1, 2)$, $c(3, 6, 2) = (1, 1, 1)$, and $c(3, 7, 2) = (2, 1)$. As a consequence $c(3, n, 2) = (1, 1, 1)$ iff $n = 6 + 4k$ for some nonnegative integer k . By Corollary 1, $\lambda(3, n, 2)$ is standard iff $n = 6 + 4k$ for some nonnegative integer k . \square

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DEPARTMENT OF MATHEMATICS, ALLEGHENY COLLEGE, MEADVILLE, PA 16335
E-mail address: mbarry@alleggheny.edu